ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/327447465

Deriving the Lagrangian Density of an Electromagnetic Field

Chapter · January 2019

DOI: 10.1007/978-3-319-91023-9_5

CITATIONS
0

READS
6,628

1 author:

Bahman Zohuri
Galaxy Advanced Engineering

486 PUBLICATIONS **1,348** CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Project Our Daily life dependency driven by renewable and nonrenewable source of energy. View project

Project I am working on cloud computation and mathematical modeling of Fuzzy logic in cloud data warehousing and datamining View project

All content following this page was uploaded by Bahman Zohuri on 30 January 2019.

The user has requested enhancement of the downloaded file.

Chapter 5
Deriving the Lagrangian Density of an Electromagnetic Field

As part of understanding relativistic particles and electromagnetic fields, we need to have some awareness of the kinematic special theory of relativity, and then turn our attention to the dynamic aspect of charged particle motion in external electromagnetic fields. Thus, in this chapter we take a Lagrangian approach to the equations of motion and deal with the total energy involved with the motion of a particle. We also introduce the Hamiltonian in relation to the total energy of particle motion; analogous to classical mechanics, it relates to the corresponding *kinetic* and *potential* energy of the particle or system of concern. The Lagrangian approach is based in electrodynamics; thus, equations of motion are presented mainly as an avenue to introduce the concept of a Lorentz invariant action to a Hamiltonian, with the definition of the canonical momentum discussed in this chapter.

5.1 Introduction

Lagrangian field theory is a formalism in classical field theory. It is the *field-theoretic* analogue of Lagrangian mechanics, which is used for discrete particles each with a finite number of degrees of freedom. Lagrangian field theory applies to continua and fields that have an infinite number of degrees of freedom.

The relativistic aspects of Hamiltonian and Lagrangian theories, which are rather abstract constructions in *classical mechanics*, have a very simple interpretation in *relativistic quantum mechanics*. They both are proportional to the number of phase changes per unit of time. The Hamiltonian runs over the time axis, whereas the Lagrangian runs over the trajectory of the moving particle, the t'-axis, as illustrated in Fig. 5.1.

The figure shows the relativistic *de Broglie wave* in a *Minkowski diagram*. The triangle represents the relation between the Lagrangian ($\mathcal{L}$) and the Hamiltonian ($\mathcal{H}$), which holds in both relativistic and non-relativistic physics; it can be written with classical mechanics notation:

© Springer International Publishing AG, part of Springer Nature 2019

B. Zohuri, *Scalar Wave Driven Energy Applications*,
https://doi.org/10.1007/978-3-319-91023-9_5

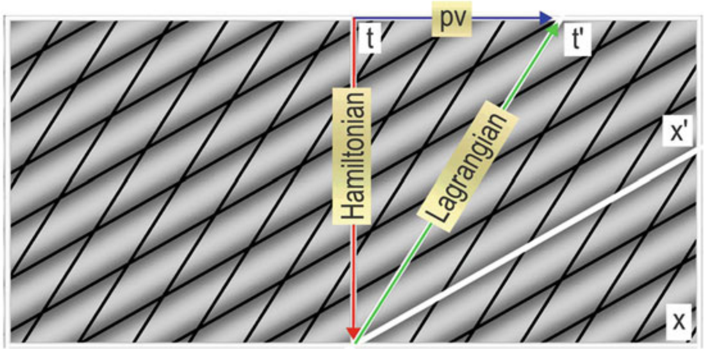


Fig. 5.1 The Hamiltonian and Lagrangian depictions scenario

$$\mathcal{L} = pv - \mathcal{H} \tag{5.1}$$

The Hamiltonian counts the phase changes per unit of time on the vertical axis, whereas the term pv counts the phase changes per unit on the horizontal axis; v is the distance traveled per unit of time, whereas p is proportional with the phase changes per unit of distance—therefore the term pv. We now can understand the classical relation with $\dot{q} = \dot{x} = v$ and write the following relation:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = p \tag{5.2}$$

Thus, for the free classical relativistic particles we have the Hamiltonian energy and the pv as follows:

$$\mathcal{H} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad pv = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.3}$$

Calculating the Lagrangian, we can see that the Hamiltonian is proportional to γ, whereas the Lagrangian is proportional to $1/\gamma$,

$$\mathcal{L} = -(\mathcal{H} - pv) = -\left[\frac{c^2 - v^2}{\sqrt{1 - \frac{v^2}{c^2}}}\right] m = -\left(\sqrt{1 - \frac{v^2}{c^2}}\right) mc^2 \tag{5.4}$$

This is what we expect from time dilation. The moving particle has fewer clock-ticks by a factor γ because of time dilation; we now check that:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial}{\partial v}\left\{-\left(\sqrt{1 - \frac{v^2}{c^2}}\right) mc^2\right\} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = p \tag{5.5}$$

We have not yet discussed potential energy, which we need to expand on. To obtain the equation for motion of a relativistic particle in a potential field, we have to add the *potential energy* term $V(q)$. In the non-relativistic case we have

$$\begin{aligned}\mathcal{H} &= T(\dot{q}) + V(q)\\ \mathcal{L} &= T(\dot{q}) - V(q)\end{aligned} \tag{5.6}$$

where $T(\dot{q}) = \frac{1}{2}mv^2$ is kinetic energy.

The relativistic Hamiltonian and Lagrangian we have discussed, however, also include rest mass energy. The *rest mass energy* can be considered part of potential energy. Kinetic part T in the relativistic case can be obtained as follows:

$$\mathcal{H} + \mathcal{L} = 2T = pv \Rightarrow T = \frac{1}{2}pv = \frac{1}{2}mv^2\left\{\sqrt{1 - \frac{v^2}{c^2}}\right\}^{-1} \approx \frac{1}{2}mv^2 \tag{5.7}$$

The approximation in this equation is valid when $v \ll c$. Using the term $\mathcal{L} = \frac{1}{2}mv^2$ in Eq. 5.5 provides the result for $p = mv$ in the non-relativistic momentum, the same as we got here.

5.2 How the Fields Transform

As part of the derivation of the Lagrangian density of electromagnetic (EM) fields, we need to have some fundamental understanding of what field transformations and field tensors are all about; these two subjects are presented in this and the following sections. What we need to get our mind around is the *general* transformation rules for EM fields so that we can answer, for example, this question:

1. Given the fields in *system S*, what are the magnetic fields in *system* S'?

To answer this question, our first guess might be that electric field $\vec{E}$ is the spatial part of one 4-vector, and the magnetic field $\vec{B}$ is the spatial part of another. If this is true, then our intuition of one person's electric field as another's magnetic field is an incorrect assumption. Thus, let us begin by making an explicit assumption that we very implicitly can express in relativistic electrodynamics—that is, charge itself is an invariant case.

Covariance	The property of a function to retain its form when the variables are linearly transformed . . . .under a covariance condition.
Invariance	In mathematics an invariance is a property held by a class of mathematical objects that remains unchanged when transformations of a certain type are applied to the objects.
Invariant	A function, quantity, or property that remains unchanged when a specified transformation is applied.

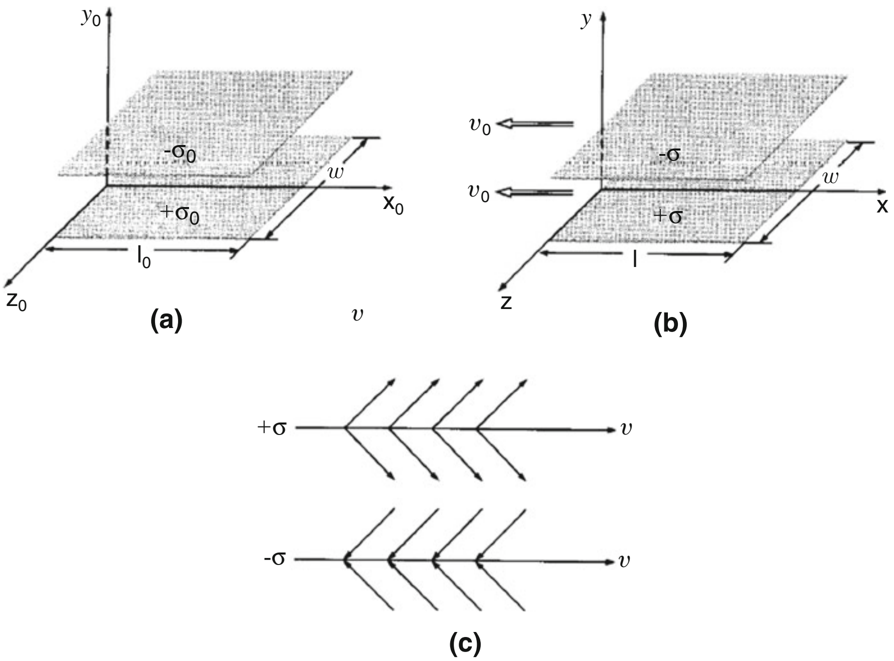


Fig. 5.2 Illustration of moving capacitor

This includes like mass, but unlike energy; the charge of the particle is a fixed number, independent of how fast the particle happens to travel from one point to another. Thus, we assume that the transformation rules are the same no matter how the fields are produced—that is, electric fields get generated by moving currents and transform the same way as those set up by stationary charges. If this is not the case, then we would need to abandon the field formulation altogether because what it is essential for a field theory is that the fields at a given point provide *all the information* one needs to know. Yet, in case of EM fields, one does not have to have extra information, electromagnetically, regarding their source [1].

Bearing the preceding argument in mind, we can consider one of the *simplest possible* electric fields, which is uniform in the region between the *two-large parallel-capacitor plates*, as shown in Fig. 5.2a, by assuming that the capacitor is at rest in S_0 and carries surface charges $\pm\sigma_0$. Then, we can write the electric field $\vec{E}_0$ in terms of *surface charge density* σ_0 in free space and $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \times \mathrm{m}^2$, this being the *permittivity of free space*, as well as a unit vector $\widehat{y}$ in the y-direction, as follows:

$$\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\widehat{j} \tag{5.8}$$

If we consider the same capacitor from system S moving to the right at speed v_0, as shown in Fig. 5.2b, however, we can see that these system plates are moving to the left, but the field takes the form of the following relationship [2]:

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\widehat{y} \tag{5.9}$$

The only difference between Eqs. 5.8 and 5.9 is the value of the surface charge σ, but we need to see whether that is the only difference. Equation 5.9 is written for a parallel plate capacitor, which came from *Gauss's Law*, whereas Gauss's Law is perfectly valid for moving charges, where this application totally relies on the symmetry issue of the problem. If the plates tilt, however, as we can see in Fig. 5.2c, is it possible to say that electric filed $\vec{E}$ still is perpendicular to the moving direction of the plate? The answer is that it does not matter, and the field between the plates is the creation of the superposition of the surface charge of $+\sigma$. The one from the $-\sigma$ nevertheless would run perpendicular to the plates that are in motion, even in a tilted condition. For the surface charge, $-\sigma$, field would orient as indicated in Fig. 5.2c; changing the sign of the charges reverses the direction of the field, and the vector sum kills off the parallel components [2].

Now we are at the point where we can consider the total charge on each plate and claim that as *invariant* and the *width* (w) is unchanged; however, the *length* (l) is *Lorentz contracted* by a factor of:

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2} \tag{5.10}$$

So that the charge per unit area is *increased* by a factor of γ_0 as indicated here:

$$\sigma = \gamma_0\sigma_0 \tag{5.11}$$

Thus, accordingly, we can say that:

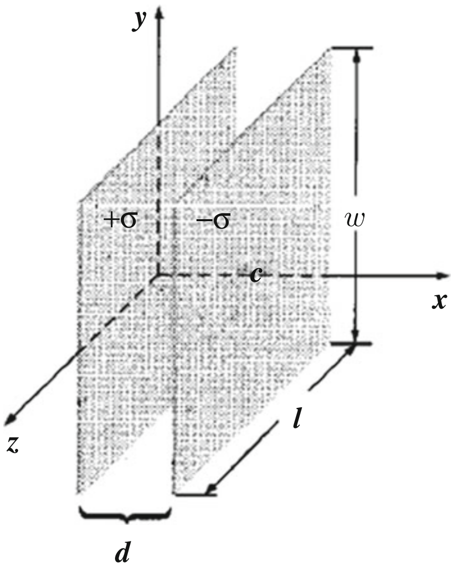
$$\vec{E}^{\perp} = \gamma_0\vec{E}_0^{\perp} \tag{5.12}$$

The symbol $\perp$ is clearly an indication of a *perpendicular* component of electric field $\vec{E}$, while symbol $\|$ is a presentation of a *parallel* component of same field in the direction of the motion of system S, considering that the capacitor lined up with the yz-plane, as shown in Fig. 5.3. At this time, it is plate separation d that is Lorentz connected, whereas l and w, consequently, σ, are the same in both frames. Because the field does not depend on d, it follows that:

$$\vec{E}_{\|} = \gamma_0\vec{E}_{\|}^{0} \tag{5.13}$$

What we have learned so far from the previous discussion we can put into perspective with the following examples given by Griffiths [2]. Here, we are going to show a simple problem of an electric field with a point charge in uniform motion.

Fig. 5.3 Two parallel charged plates



Example 5.1 A point charge q is at the origin in system S_0.

Question What is the electric field of this same charge in system S, which moves to the right at speed υ_0 relative to system S_0?

Solution In system S_0, the field is:

$$\vec{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0^2}\hat{r}_0$$

$$\begin{cases} E_{x0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qz_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \\ E_{y0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qx_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \\ E_{z0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qy_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \end{cases}$$

From the transformation rules of Eqs. 5.12 and 5.13, we can state that:

$$\begin{cases} E_x = E_{x0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qx_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \\ E_y = \gamma_0 E_{y0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma_0 q y_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \\ E_z = \gamma_0 E_{z0} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma_0 q z_0}{\left(x_0^2 + y_0^2 + z_0^2\right)^{3/2}} \end{cases}$$

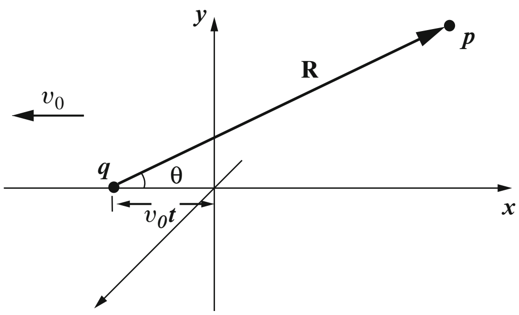


Fig. 5.4 Illustration of point charge in system S_0

The expression is still in terms of the system S_0 coordinates (x_0, y_0, z_0) of the field point (P), and preferably we need to write them in terms of the system S coordinate of point (P). From the Lorentz transformations, or actually, the inverse transformations point of view, we can write:

$$\begin{cases} x_0 = \gamma_0(x + v_0 t) = \gamma_0 R_x \\ y_0 = \quad y \quad = R_y \\ z_0 = \quad z \quad = R_z \end{cases}$$

where $\vec{R}$ is the vector from charge q to point P, as depicted in Fig. 5.4. Thus, we can write:

$$\begin{aligned} \vec{E} &= \frac{1}{4\pi\varepsilon_0} \frac{\gamma_0 q \vec{R}}{\left[\gamma_0^2 R^2 \cos^2(\theta) + R^2 \sin^2(\theta)\right]^{3/2}} \\ &= \frac{1}{4\pi\varepsilon_0} \frac{q\left(1 - v_0^2/c^2\right)}{\left[1 - \left(v_0^2/c^2\right)\sin^2(\theta)\right]^{3/2}} \frac{\widehat{R}}{R^2} \end{aligned} \tag{5.14}$$

This equation then is identified as the field charge in uniform motion; thus, we get the same result if we use a retarded potential equation, known as *Lienard-Wiechert potential* for point charges if they move on a specific trajectory [2].

This is a significant and efficient derivation, and shows that the field points away from the instantaneous as opposed to the retarded position of the charge; E_x gets a factor of γ_0 from the *Lorentz transformation* of the *coordinates*, and E_y as well as E_z pick up their factors from the transformation of the *field*. It is the balancing of these two γ_0s that leaves $\vec{E}$ parallel to $\vec{R}$.

Nonetheless, Eqs. 5.12 and 5.13 are not the most general transformation laws that we can express because we start with system S_0 in which the charges were at rest and where, consequently, there was no magnetic field. For us to derive the *general* rule, what we need to start with is out in a system with both electric and magnetic fields simultaneously. For this purpose, S itself will serve very nicely.

In addition to the *electric* filed,

$$E_y = \frac{\sigma}{\varepsilon_0} \tag{5.15}$$

there exists a *magnetic* field because of the surface currents, as shown earlier in Fig. 5.2b:

$$\vec{K}_{\mp} = \mp\sigma\upsilon_0\widehat{x} \tag{5.16}$$

Using the right-hand side (RHS) rule, as it was mentioned in Chap. 1 of this book, this field points in the negative z-direction and its magnitude is given by Ampère's Law as:

$$B_z = -\mu_0\sigma\upsilon_0 \tag{5.17}$$

Using a figure, such as depicted Fig. 5.5, for a *third* system $\bar{S}$, traveling to the right with speed υ relative to S, the field would be:

$$\bar{E}_y = \frac{\overline{\sigma}}{\varepsilon_0}\bar{B}_z = -\mu_0\overline{\sigma\upsilon} \tag{5.18}$$

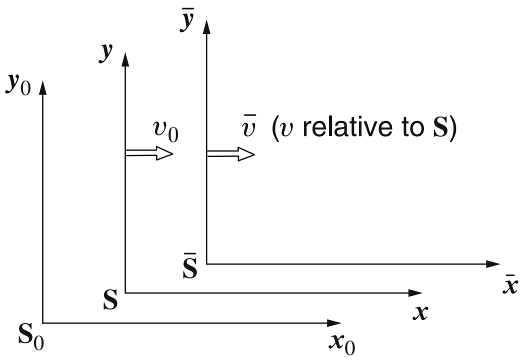
where $\overline{\upsilon}$ is the velocity of system $\bar{S}$ relative to system S_0, and it can be written in the following form:

$$\begin{aligned}\overline{\upsilon} &= \frac{\upsilon + \upsilon_0}{1 + \upsilon\upsilon_0/c^2} \\ \overline{\gamma} &= \frac{1}{\sqrt{1 - \overline{\upsilon}^2/c^2}}\end{aligned} \tag{5.19}$$

and

$$\overline{\sigma} = \overline{\gamma}\sigma_0 \tag{5.20}$$

Fig. 5.5 Illustration of systems S_0, S_0, S_0 and their motions



The only thing that remains is to express electric filed $\vec{\bar{E}}$ and magnetic field $\vec{\bar{B}}$, as shown in Eq. 5.18, in terms of $\vec{E}$ and $\vec{B}$ (see Eqs. 5.15 and 5.17). In view of Eqs. 5.11 and 5.20, we have the following:

$$\begin{aligned} \bar{E}_y &= \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0} \\ \bar{B}_z &= -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v} \end{aligned} \tag{5.21}$$

With a little manipulation of algebra, we find the following result:

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \bar{v}^2/c^2}} = \frac{1 + vv_0/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(1 + \frac{vv_0}{c^2}\right) \tag{5.22}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{5.23}$$

Thus, we can write:

$$\bar{E}_y = \gamma\left(1 + \frac{vv_0}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(E_y - \frac{v}{c^2\varepsilon_0\mu_0}B_z\right) \tag{5.24}$$

whereas

$$\begin{aligned} \bar{B}_2 &= -\gamma\left(1 + \frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v + v_0}{1 + vv_0/c^2}\right) \\ &= \gamma\left(B_z - \mu_0\varepsilon_0 v E_y\right) \end{aligned} \tag{5.25}$$

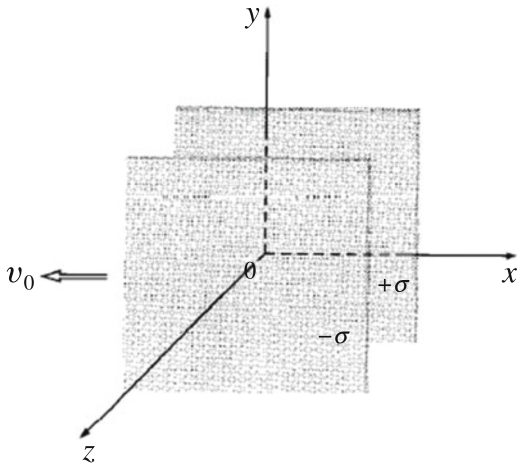
Or, because $\mu_0\varepsilon_0 = 1/c^2$, then Eqs. 5.24 and 5.25 reduce to the following form:

$$\begin{cases} \bar{E}_y = \gamma\left(E_y - vB_z\right) \\ \bar{B}_z = \gamma\left(B_z - \dfrac{v}{c^2}E_y\right) \end{cases} \tag{5.26}$$

This equation indicates how E_y and B_z transform—for E_z and B_y simply align the same capacitor parallel to the xy-plane instead of xz-plane, as illustrated in Fig. 5.6. Thus, the fields in system S are written:

$$\begin{aligned} E_z &= \frac{\sigma}{\varepsilon_0} \\ B_y &= \mu_0\sigma v_0 \end{aligned} \tag{5.27}$$

Fig. 5.6 Two capacitors parallel to the *xy*-plane



Using the RHS rule of Chap. 1 again, we get B_y. The rest of the argument is identical—everywhere we had E_y before reads E_z, and everywhere that we had B_z reads $-B_y$; thus, we can write:

$$\begin{cases} \bar{E}_z = \gamma\left(E_z + vB_y\right) \\ \bar{B}_y = \gamma\left(B_y + \dfrac{v}{c^2}E_z\right) \end{cases} \tag{5.28}$$

As for the *x*-components, we already have seen, by orienting the capacitor parallel to the *yz*-plane, that:

$$\bar{E}_x = E_x \tag{5.29}$$

Because in this case there is no accompanying magnetic field, we cannot conclude and deduce the transformation rule for B_x. Nevertheless, another configuration, such as the one in Fig. 5.7, which is an illustration of a long *solenoid* aligned parallel to the *x*-axis and at rest in system *S*. The magnetic field within the coils is then [2]:

$$B_x = \mu_0 nI \tag{5.30}$$

where *n* is the number of turns per unit length and *I* is the current in the coil. In system $\bar{S}$ the length contracts, so *n increases*:

$$\bar{n} = \gamma n \tag{5.31}$$

On the other hand, time *dilates*. Thus, the system *S* clock that rides along with the solenoid runs slow, so the current (i.e., change *per unit time*) in $\bar{S}$ is given by:

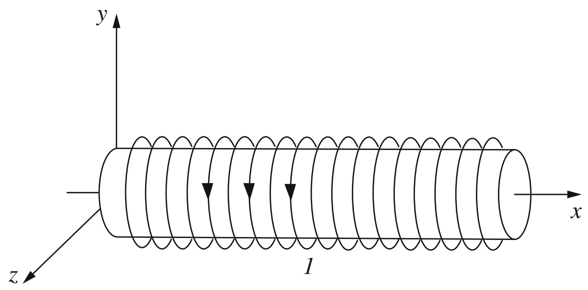


Fig. 5.7 Depiction of long solenoid aligned parallel to the *x*-axis

$$\bar{I} = \frac{1}{\gamma} I \tag{5.32}$$

The two factors of γ exactly cancel, and we conclude that:

$$\bar{B}_x = B_x \tag{5.33}$$

Like $\vec{E}$, component $\vec{B}$, which is *parallel* to the motion, is unchanged [1].

Let us now collect all the components of transformation in order to have a complete set of transformation rules:

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x & \bar{E}_y = \gamma\left(E_y - vB_z\right) & \bar{E}_z = \gamma\left(E_z + vB_y\right) \\ \bar{B}_x = B_x & \bar{B}_y = \gamma\left(B_y + \dfrac{v}{c^2}E_z\right) & \bar{B}_z = \gamma\left(B_z - \dfrac{v}{c^2}E_y\right) \end{array}} \tag{5.34}$$

The following are two special cases that we need to pay close attention to:

1. If $\vec{B} = 0$ in system S, then we can write:

$$\bar{\vec{B}} = \gamma \frac{v}{c^2}\left(E_z\hat{y} - E_y\hat{z}\right) = \frac{v}{c^2}\left(\bar{E}_z\hat{y} - \bar{E}_y\hat{z}\right) \tag{5.35}$$

or, since $\vec{v} = v\hat{x}$, then

$$\bar{\vec{B}} = -\frac{1}{c^2}\left(\vec{v} \times \bar{\vec{E}}\right) \tag{5.36}$$

2. If $\vec{E} = 0$ in system S, then

$$\bar{\vec{E}} = -\gamma v\left(B_z\hat{y} - B_y\hat{z}\right) = -v\left(\bar{B}_z\hat{y} - \bar{B}_y\hat{z}\right) \tag{5.37}$$

or

$$\bar{\vec{E}} = \vec{v} \times \vec{B} \tag{5.38}$$

5.3 The Field Tensor

As part of electrodynamics and relativity, we can expand our knowledge into the *field tensor*, where the classical electrodynamics *already* is consistent with *special relativity* that can be extended and be applied to *Maxwell's equations* and the *Lorentz force law* It is possible, legitimately, in any inertial system, even in an *invariant Lorentz transformation*, to show how the field transforms. See Appendix B for more details [2].

To understand what we discussed as part of developing Eq. 5.6 and the scalar $\left(\|\vec{E}\|^2 - c^2\|\vec{B}\|^2\right)$, one of the two *Lorentz invariants*, we consider the following tensor field analyses; refer to Appendix B for more essential information and derivations. Still, bear in mind that a 4-vector transforms by the rule shown in the following equation:

$$a^{\mu'} = \Lambda^{\mu}_{\nu} a^{\nu} \tag{5.39}$$

Note that the summation over v implies that a Λ is the *Lorentz transformation matrix*. If S' is moving in the x-direction at speed $\vec{v}$, then Λ has the form,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.40}$$

Λ^{μ}_{ν} is the entry in row μ and column v. A second-rank tensor is an object with *two* indices that transforms with *two* factors of Λ, which is one for each index as demonstrated in the following equation, where we have:

$$t^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma} \tag{5.41}$$

A tensor in four dimensions has $4 \times 4 = 16$ components, which we can display in a 4×4 matrix array:

$$t^{\mu\nu} = \begin{bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{bmatrix} \tag{5.42}$$

But, the 16 elements need not all to be different. For instance, a *symmetric* tensor has the property as follows:

$$t^{\mu\nu} = t^{\nu\mu} \quad \text{(Symmetric Tensor)} \tag{5.43}$$

In this case, there are 10 distinct components and 6 out of 16 represent the repeats ($t^{01}=t^{10}$, $t^{02}=t^{20}$, $t^{03}=t^{30}$, $t^{12}=t^{21}$, $t^{13}=t^{31}$, and $t^{23}=t^{32}$). Similarly, and *antisymmetric* tensor obeys the following relation as well:

$$t^{\mu\nu}=-t^{\nu\mu}\quad(\text{Antisymmetric Tensor})\tag{5.44}$$

Such an object has just 6 distinct elements of the original 16 components, where 6 are repeats of the same ones as before, only this time with a minus sign in front of each element, and 4 of them are zeros; they are (t^{00}, t^{11}, t^{22}, and t^{33}). Thus, the general form of an antisymmetric tensor has the following format:

$$t^{\mu\nu}=\begin{bmatrix}0 & t^{01} & t^{02} & t^{03}\\ -t^{10} & 0 & t^{12} & t^{13}\\ -t^{20} & -t^{21} & 0 & t^{23}\\ -t^{30} & -t^{31} & -t^{32} & 0\end{bmatrix}\tag{5.45}$$

Now let us see how the transformation of the rule defined by Eq. 5.41 works for the 6 distinct components of an anti-symmetric tensor. Thus, starting with component $-t^{10}$, we have the following relation:

$$-t^{01}=\Lambda_{\lambda}^{0}\Lambda_{\sigma}^{1}t^{\lambda\sigma}\tag{5.46}$$

According to Eq. 5.40, $\Lambda_{\lambda}^{0}=0$; however, unless $\lambda=0$ or 1, and $\Lambda_{\sigma}^{1}=0$ unless $\sigma=0$ or 1. Thus, there are four terms in the sum:

$$-t^{01}=\Lambda_0^0\Lambda_0^1t^{00}+\Lambda_0^0\Lambda_1^1t^{01}+\Lambda_1^0\Lambda_0^1t^{10}+\Lambda_1^0\Lambda_1^1t^{11}\tag{5.47}$$

On the other hand, $t^{00}=t^{11}=0$, while $t^{01}=-t^{10}$, then we can write the following relations:

$$-t^{01}=\left(\Lambda_0^0\Lambda_1^1-\Lambda_1^0\Lambda_0^1\right)t^{01}=\left[\gamma^2-(\gamma\beta)^2\right]t^{01}=t^{01}\tag{5.48}$$

Working out the other components, similarly, we obtain the following results:

$$\begin{cases}-t^{01}=t^{01} & -t^{02}=\gamma(t^{02}-\beta t^{12}) & -t^{03}=\gamma(t^{03}-\beta t^{31})\\ -t^{23}=t^{23} & -t^{31}=\gamma(t^{31}-\beta t^{03}) & -t^{12}=\gamma(t^{12}-\beta t^{02})\end{cases}\tag{5.49}$$

These are precisely the rules we can derive on physical grounds for the EM field in Sect. 5.2 [1].

In fact, we can construct *field tensor* $F^{\mu\nu}$ by direct comparison as:

$$\begin{array}{lll}F^{01}=\dfrac{E_x}{c}, & F^{02}=\dfrac{E_y}{c}, & F^{03}=\dfrac{E_z}{c}\\ F^{12}=B_z, & F^{31}=B_y, & F^{23}=B_x\end{array}\tag{5.50}$$

Written in Eq. 5.50 as a matrix array, we can produce the following:

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_x/c & B_y & -B_x & 0 \end{bmatrix} \tag{5.51}$$

With the results obtained in Eq. 5.51, relativity completes and makes a perfect job by Oersted, combining the electric and magnetic fields into a single entity as $F^{\mu\nu}$. Note that some authors prefer the other notation convention—$F^{01} = E_x$, $F^{12} = cB_z$, and so on—and some use the opposite signs. Accordingly, most of the equations from here on will have a little different appearance, depending on the text you are reading.

If we follow the argument that yields Eq. 5.51 results, with some cautious care, we can notice that there is a *different* way of relating an anti-symmetric tensor with electric field $\vec{E}$ and magnetic field $\vec{B}$. Instead of comparing the first line of Eq. 5.51 with the first line of Eq. 5.49, and the second with the second, we can relate the first line of Eq. 5.51 to the *second* line of Eq. 5.49 and vice versa. This leads to and results in the *dual tensor*, $G^{\mu\nu}$, as follows:

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & -E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix} \tag{5.52}$$

$G^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ by the substitution of $\vec{E}/c \rightarrow \vec{B}$ and $\vec{B} \rightarrow \vec{E}/c$. Notice that this operation leaves Eq. 5.51 unchanged; this is why both tensors generate the correct transformation rules for $\vec{E}$ and $\vec{B}$.

5.4 The Electromagnetic Field Tensor

The transformation of electric and magnetic fields under a Lorentz boost was established even before Einstein developed the theory of relativity. We know the $\vec{E}$-fields can transform into $\vec{B}$-fields and vice versa. For example, a point charge at rest gives an *electric* field. If we boost to a frame in which the charge is moving, there is an *electric* and a *magnetic* field. This means that the $\vec{E}$-field cannot be a *Lorentz vector*. We need to put the electric and magnetic fields into one *tensor object* to properly handle a *Lorentz transformation* and to write the equations in a covariant way.

The simplest way, and the correct way, to do this is to make the electric and magnetic fields components of a rank 2 anti-symmetric tensor as follows:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \tag{5.53}$$

The fields can be written simply in terms of the vector potential, which is Lorentz vector $A_\mu = \left(\vec{A}, i\phi\right)$, thus Eq. 5.53 can be expressed in the following form:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{5.54}$$

Note that this is automatically anti-symmetric under the interchange of the indices. As before, the first two source-less *Maxwell equations* are automatically satisfied for fields derived from potential. We may write the other two Maxwell equations in terms of 4-vector $j_\mu = \left(\vec{j}, ic\rho\right)$ in the following form:

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{j_\mu}{c} \tag{5.55}$$

Comparing Eqs. 5.54 and 5.55, we can write the following:

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{j_\mu}{c} \tag{5.56}$$

Of course, we have not yet quantized the theory in Eq. 5.56.

For some peace of mind, let us verify a few terms in the equations. Clearly, all the diagonal terms in the field tensor are zero by anti-symmetry. Taking some examples of off-diagonal terms in the field tensor, check the old definition of the fields in terms of potential:

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
\vec{E} &= -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\
F_{12} &= \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \left(\vec{\nabla} \times \vec{A}\right)_z = B_z \\
F_{13} &= \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} = \left(\vec{\nabla} \times \vec{A}\right)_y = -B_y \\
F_{4i} &= \frac{\partial A_i}{\partial x_4} - \frac{\partial A_4}{\partial x_i} = \frac{1}{ic}\frac{\partial A_i}{\partial t} - \frac{\partial (i\phi)}{\partial x_i} = -i\left(\frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i}\right) = -i\left(\frac{\partial \phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right)
\end{aligned} \tag{5.57}$$

Let us also check what the Maxwell equation says for the last row in the tensor, as follows:

$$\begin{aligned}
&\frac{\partial F_{4v}}{\partial x_v} = \frac{j_4}{c} \\
&\frac{\partial F_{4i}}{\partial x_i} = \frac{ic\rho}{c} \\
&\frac{\partial (iE_i)}{\partial x_i} = i\rho \\
&\frac{\partial E_i}{\partial x_i} = \rho \\
&\vec{\nabla} \cdot \vec{E} = \rho
\end{aligned} \tag{5.58}$$

We will not bother to check the Lorentz transformation of the field here.

5.5 The Lagrangian and Hamiltonian for Electromagnetic Fields

There are not many ways to make a *scalar Lagrangian* from the field tensor. We already, from Eq. 5.55, know that

$$\frac{\partial F_{\mu v}}{\partial x_v} = \frac{j_\mu}{c} \tag{5.59}$$

In addition, we need to make our Lagrangian out of the field, not just the current again; x_μ cannot appear explicitly because it violates symmetries. Also, we want a linear equation, so higher powers of the field should not occur. Term $mA_\mu A^\mu$ is a mass term and would cause fields to fall off faster than $1/r$ (i.e., we later show how the term $1/r$ comes into play). Thus, the only reasonable choice is:

$$F_{\mu v} F^{\mu v} = 2\left(B^2 - E^2\right) \tag{5.60}$$

One might consider the following relationship:

$$e_{\mu v} \lambda_\sigma F_{\mu v} F_{\lambda \sigma} = \vec{B} \cdot \vec{E} \tag{5.61}$$

But that is a pseudo-scalar, not a scalar completely—that is, it changes its sign under a parity transformation. The EM interaction is known to conserve parity, so this is not a real option. As with the scalar field, we need to add an interaction with a source term. Of course, we know electromagnetism well, so finding the right Lagrangian is not really guess work. The source of the field is the vector $\vec{j}_\mu$; thus, the simple scalar that we can write is $j_\mu A^\mu$.

The Lagrangian for *classical electricity and magnetism* we will try is:

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}\vec{j}_\mu A^\mu \tag{5.62}$$

In working with this Lagrangian, we will treat each component of A as an independent field.

The next step is to check what the *Euler-Lagrange equation* provides:

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial A^\mu/\partial x_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \tag{5.63}$$

where

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}j_\mu A^\mu = -\frac{1}{4}\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) \tag{5.64}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial A^\mu/\partial x_\nu)} &= -\frac{1}{4}\frac{\partial}{\partial(\partial A^\mu/\partial x_\nu)}\left(\frac{\partial A^\sigma}{\partial x_\lambda} - \frac{\partial A^\lambda}{\partial x_\sigma}\right)\left(\frac{\partial A^\sigma}{\partial x_\lambda} - \frac{\partial A^\lambda}{\partial x_\sigma}\right)\\
&= -\frac{1}{4}\frac{\partial}{\partial(\partial A^\mu/\partial x_\nu)}\left(2\frac{\partial A^\sigma}{\partial x_\lambda}\frac{\partial A^\sigma}{\partial x_\lambda} - 2\frac{\partial A^\sigma}{\partial x_\lambda}\frac{\partial A^\lambda}{\partial x_\sigma}\right)\\
&= -\frac{1}{4}4\left(\frac{\partial A^\lambda}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu}\right)\\
&= -F_{\nu\mu} = F_{\mu\nu}
\end{aligned} \tag{5.65}$$

$$\begin{cases}
\dfrac{\partial}{\partial x_\nu}F^{\mu\nu} - \dfrac{\partial \mathcal{L}}{\partial A_\mu} = 0\\
\dfrac{\partial}{\partial x_\nu}F^{\mu\nu} - \dfrac{j_\mu}{c} = 0\\
\dfrac{\partial}{\partial x_\nu}F^{\mu\nu} = \dfrac{j_\mu}{c}
\end{cases} \tag{5.66}$$

Now that we have four independent components of A^μ as autonomous fields, we have four equations or one 4-vector equation. The Euler-Lagrange equation gets us back to Maxwell's equations with this choice of the Lagrangian. Clearly, this justifies the choice of $\mathcal{L}$.

It is important to emphasize that we have a Lagrangian-based, formal classical field theory for electricity and magnetism, which has the four components of the 4-vector potential as the independent fields. We could not treat each component of $F_{\mu\nu}$ as independent because they obviously are correlated. We could have tried using the six independent components of the anti-symmetric tensor, but it would not have given the correct answer. Using the 4-vector potentials as the fields does give the correct answer. Electricity and magnetism is a theory of 4-vector field A^μ.

We also can calculate the free field Hamiltonian density—that is, the *Hamiltonian density* in regions with no source term. We use the standard definition of the Hamiltonian in terms of the Lagrangian:

$$\mathcal{H} = \left(\frac{\partial \mathcal{L}}{\partial(\partial A^{\mu}/\partial dt)}\right)\frac{\partial A^{\mu}}{\partial dt} - \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial(\partial A^{\mu}/\partial x_4)}\right)\frac{\partial A^{\mu}}{\partial x_4} - \mathcal{L} \tag{5.67}$$

We calculated before that:

$$\frac{\partial \mathcal{L}}{\partial(\partial A^{\mu}/\partial x_{\nu})} = F_{\mu\nu} \tag{5.68}$$

which we can use to get:

$$\frac{\partial \mathcal{L}}{\partial(\partial A^{\mu}/\partial x_4)} = F_{\mu 4} \tag{5.69}$$

Thus, the free field Hamiltonian density becomes:

$$\begin{aligned} \mathcal{H} &= \left(F_{\mu 4}\right)\frac{\partial A^{\mu}}{\partial x_4} - \mathcal{L} \\ &= F_{\mu 4}\frac{\partial A^{\mu}}{\partial x_4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{5.70}$$

Therefore,

$$\boxed{\mathcal{H} = F_{\mu 4}\frac{\partial A^{\mu}}{\partial x_4} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \tag{5.71}$$

We will use this once we have written the radiation field in a convenient form. In the meantime, we can check what this gives us in general in a region with no sources:

$$\begin{aligned} \mathcal{H} &= F_{\mu 4}\left(F_{4\varpi} + \frac{\partial A^{4}}{\partial x_{\mu}}\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -F_{\mu 4}\left(F_{4\mu} + \frac{\partial A^{4}}{\partial x_{\mu}}\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -F_{\mu 4}F_{4\mu} - F_{\mu 4} + \frac{\partial A^{4}}{\partial x_{\mu}} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= E^{2} - F_{4i}\frac{\partial A^{4}}{\partial x_{i}} + \frac{1}{2}\left(B^{2} - E^{2}\right) \\ &= \frac{1}{2}\left(E^{2} + B^{2}\right) - iE_{i}\frac{\partial(i\phi)}{\partial x_{i}} \\ &= \frac{1}{2}\left(E^{2} + B^{2}\right) + E_{i}\frac{\partial \phi}{\partial x_{i}} \end{aligned} \tag{5.72}$$

If we integrate the last term by parts (and the fields fall to zero at infinity), then that term contains a $\vec{\nabla} \cdot \vec{E}$, which is zero with no sources in the region. We therefore can drop it and are left with:

$$\mathcal{H} = \frac{1}{2}\left(E^2 + B^2\right) \tag{5.73}$$

This is the result we expected: the energy density and an EM field. (Remember the fields have been decreased by a factor of $\sqrt{4\pi}$ compared to centimeter-gram-second, CGS, units.)

We will study the interaction between electrons and the EM field with the *Dirac equation* later. Until then, the Hamiltonian used for non-relativistic quantum mechanics will be sufficient. We have derived the Lorentz force law from the Hamiltonian.

5.6 Introduction to Lagrangian Density

In considering the *special relativistic EM field*, we understand that assuming a *Lagrangian density* of the form results in:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{c}\vec{j}_\mu A^\mu \tag{5.74}$$

and following the *Euler-Lagrange equation* recovers Maxwell's equation. In this equation $F^{\mu\nu}$ is the Maxwell field tensor and c is the speed of light, $c^2 = 1/\varepsilon\mu$ (not necessarily in a vacuum). The four-current is $\vec{j}_\mu = \left(\rho c, \vec{j}\right)$; the four-potential is $\vec{A}_\mu = \left(\phi/c, \vec{A}\right)$; the Compton wave number for a photon with mass (m) takes the form $k = 2\pi mc/h$; and h is *Plank's constant.*

In the following we will prove that a compatible *Lagrangian density* for the EM field in empty space is,

$$\mathcal{L}_{em} = \varepsilon_0\left\{\frac{\|\vec{E}\|^2 - c^2\|\vec{B}\|^2}{2}\right\} - \rho\phi + \vec{j}\cdot\vec{A} \tag{5.75}$$

which will be proof of Eq. 5.74. In following section, we take a very general approach to the more *classical electrodynamics* of Lagrangian density for the EM field.

Equation 5.75 is the *Euler-Lagrange equation* produced from this Lagrangian and are Maxwell's equations for the EM field. Lagrangian density is derived by a trial-and-error procedure, not by guessing, and one also can find the Lagrangian in a more difficult and complicated case. (See the proof provided in Law [3], where the interaction between a moving mirror and radiation pressure is a Hamiltonian formulation.)

Maxwell’s differential equations of the EM field in empty space are:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.76a}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{5.76b}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{5.76c}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.76d}$$

where $\vec{E}$ is the electric field intensity vector, $\vec{B}$ is the magnetic-flux density vector, ρ is the electric charge density, and $\vec{j}$ is the electric current-density vector. All quantities are functions of the three space coordinates $(x_1, x_2, x_3) \equiv (x, y, z)$ and time $t = x_4$ for time coordinates of the fourth dimension.

From Eq. 5.76d, magnetic-flux vector $\vec{B}$ may be expressed as the *curl* of vector potential $\vec{A}$ and is written as follows:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{5.77}$$

In addition, from Eqs. 5.77 and 5.76a, we can conclude that:

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{5.78}$$

So the parentheses term may be expressed as the gradient of a scalar function:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \tag{5.79a}$$

Equation 5.78 can be rearranged as follows as well:

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \tag{5.79b}$$

Therefore, the six scalar variables, the components of vectors $\vec{E}$ and $\vec{B}$, can be expressed as functions of four *scalar variables*, the scalar potential ϕ, and three components of vector potential $\vec{A}$. Inserting the expressions of $\vec{E}$ and $\vec{B}$, Eqs. 5.77, 5.79a, and 5.79b, respectively, into Eqs. 5.76b and 5.76c, we obtain the following result:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla \phi - \frac{\partial \vec{A}}{\partial t} \right) \tag{5.80}$$

and

$$\boxed{-\nabla^2\phi - \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) = \frac{\rho}{\varepsilon_0}} \tag{5.81}$$

Given that the vector identity is of this form:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} \tag{5.82}$$

Per Eqs. 5.82, and 5.80, we find the following result:

$$\boxed{\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0\vec{j}} \tag{5.83}$$

5.7 Euler-Lagrange Equation of the Electromagnetic Field

Now the main goal and task is to find a Lagrangian density $\mathcal{L}$ function of the four "field coordinates" and their first-order derivatives as follows:

$$\mathcal{L} = \mathcal{L}\left(\eta_j, \dot{\eta}_j, \vec{\nabla}\eta_j\right) \quad j = 1,2,3 \tag{5.84}$$

such that the four scalar EM fields, Eqs. 5.81 and 5.83, are derived from the Lagrange equations:

$$\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial\eta_j}{\partial t}\right)}\right] + \sum_{k=1}^{k=3}\frac{\partial}{\partial x_k}\left[\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial\eta_j}{\partial x_k}\right)}\right] - \frac{\partial\mathcal{L}}{\partial\eta_j} = 0 \quad j = 1,2,3 \tag{5.85}$$

This can be simplified in notation to:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}_j}\right) + \vec{\nabla}\cdot\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla\eta_j\right)}\right] - \frac{\partial\mathcal{L}}{\partial\eta_j} = 0 \quad j = 1,2,3} \tag{5.86}$$

Here the *Lagrangian density* $\mathcal{L}$ is a function of the following:

1. The four "field coordinates"

$$\eta_1 = A_1(x_1, x_2, x_3, t) \tag{5.87a}$$

$$\eta_2 = A_2(x_1, x_2, x_3, t) \tag{5.87b}$$

$$\eta_3 = A_3(x_1, x_2, x_3, t) \tag{5.87c}$$

$$\eta_4 = A_4(x_1, x_2, x_3, t) \tag{5.87d}$$

2. Their time derivatives

$$\dot{\eta}_1 \equiv \frac{\partial \eta_1}{\partial t} = \frac{\partial A_1}{\partial t} \equiv \dot{A}_1 \tag{5.88a}$$

$$\dot{\eta}_2 \equiv \frac{\partial \eta_2}{\partial t} = \frac{\partial A_2}{\partial t} \equiv \dot{A}_2 \tag{5.88b}$$

$$\dot{\eta}_3 \equiv \frac{\partial \eta_3}{\partial t} = \frac{\partial A_3}{\partial t} \equiv \dot{A}_3 \tag{5.88c}$$

$$\dot{\eta}_4 \equiv \frac{\partial \eta_4}{\partial t} = \frac{\partial \phi}{\partial t} \equiv \dot{\phi} \tag{5.88d}$$

3. Their gradients

$$\nabla\eta_1 = \nabla \mathrm{A}_1, \quad \nabla\eta_2 = \nabla \mathrm{A}_2, \quad \nabla\eta_3 = \nabla \mathrm{A}_3, \quad \nabla\eta_4 = \nabla\phi \tag{5.89}$$

We express Eqs. 5.81 and 5.83 in forms that are similar to Lagrange Eq. 5.86:

$$\boxed{\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right) + \vec{\nabla}\cdot(\nabla\phi) - \left(-\frac{\rho}{\varepsilon_0}\right) = 0} \tag{5.90}$$

Finally, we can write:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial A_k}{\partial t} + \frac{\partial \phi}{\partial x_k}\right) + \nabla\cdot\left[c^2\left(\frac{\partial \vec{A}}{\partial x_k} - \nabla A_k\right)\right] - \frac{j_k}{\varepsilon_0} = 0} \tag{5.91}$$

The Lagrange Eq. 5.86 for $j = 4$—that is, for $\eta_4 = \phi$ is given as:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \vec{\nabla}\cdot\left[\frac{\partial \mathcal{L}}{\partial(\nabla\phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{5.92}$$

Comparing Eqs. 5.90 and 5.2, we note that the first could be derived from the second if,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \nabla\cdot\vec{A}, \quad \frac{\partial \mathcal{L}}{\partial(\nabla\phi)} = \nabla\phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\rho}{\varepsilon_0} \tag{5.93}$$

so that Lagrangian density $\mathcal{L}$ must contain, respectively, the terms:

$$\mathcal{L}_{\alpha_1} \equiv \left(\nabla \cdot \vec{A}\right)\dot{\phi}, \quad \mathcal{L}_{\alpha_2} \equiv \frac{1}{2}\|\nabla\phi\|^2, \quad \mathcal{L}_{\alpha_3} \equiv -\frac{\rho\phi}{\varepsilon_0} \tag{5.94}$$

Consequently, their sum becomes:

$$\mathcal{L}_\alpha = \mathcal{L}_{\alpha_1} + \mathcal{L}_\alpha + \mathcal{L}_{\alpha_3} = \left(\vec{\nabla} \cdot \vec{A}\right)\dot{\phi} + \frac{1}{2}\|\nabla\phi\|^2 - \frac{\rho\phi}{\varepsilon_0} \tag{5.95}$$

We suppose that an appropriate Lagrangian density $\mathcal{L}$ would be of the form:

$$\mathcal{L} = \mathcal{L}_\alpha + \mathcal{L}_\beta \tag{5.96}$$

and because $\mathcal{L}_\alpha$ produces Eq. 5.90, we expect that $\mathcal{L}_\beta$—to be determined—will produce Eq. 5.91. This expectation would be correct if Eqs. 5.90 and 5.91 were decoupled; for example, if the first contains only ϕ-terms and the second contains only $\vec{A}$-terms. But, here this is not the case: $\mathcal{L}_\alpha$ containing $\vec{A}$-terms would participate to the production of Eq. 5.91; moreover, $\mathcal{L}_\beta$ would participate in the production of Eq. 5.90. It is possible that this could mutually destroy the production of the equations as we had expected. Nonetheless, here we follow a trial-and-error procedure, which will direct us to the correct answer, as we can see in the following.

Now Lagrangian Eq. 5.86 for $j = k = 1, 2, 3$—that is, for $\eta_k = A_k$—is:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{A}_k}\right) + \nabla \cdot \left[\frac{\partial \mathcal{L}}{\partial(\nabla A_k)}\right] - \frac{\partial \mathcal{L}}{\partial A_k} = 0 \tag{5.97}$$

Comparing Eqs. 5.91 and 5.97, we note that the first could be derived from the second if,

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_k} = \dot{A}_k + \frac{\partial \phi}{\partial x_k}, \quad \frac{\partial \mathcal{L}}{\partial(\nabla A_k)} = c^2\left(\frac{\partial \vec{A}}{\partial x_k} - \nabla A_k\right), \quad \frac{\partial \mathcal{L}}{\partial A_k} = \frac{j_k}{\varepsilon_0} \tag{5.98}$$

From the first part of Eq. 5.98 the $\mathcal{L}_\beta$ of Lagrange density $\mathcal{L}$ must contain the following terms:

$$\frac{1}{2}\|\dot{A}_k\|^2 + \frac{\partial \phi}{\partial x_k}\dot{A}_k, \quad k = 1, 2, 3 \tag{5.99}$$

So their sum with respect to k is:

$$\mathcal{L}_{\beta_1} \equiv \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \nabla\phi \cdot \dot{\vec{A}} \tag{5.100}$$

From the second part in Eq. 5.98, the $\mathcal{L}_\beta$ of Lagrange density $\mathcal{L}$ must contain the terms as follows:

$$\frac{1}{2}c^2\left[\frac{\partial \vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2\right] \quad k = 1, 2, 3 \tag{5.101}$$

So their sum with respect to k turns into this format:

$$\mathcal{L}_{\beta_2} \equiv \frac{1}{2}c^2 \sum_{k=1}^{k=3} \left[\frac{\partial \vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2 \right] \tag{5.102}$$

From the third part in Eq. 5.98, the $\mathcal{L}_\beta$ of Lagrange density $\mathcal{L}$ must contain the terms:

$$\frac{j_k A_k}{\varepsilon_0} \quad k = 1, 2, 3 \tag{5.103}$$

Therefore, their sum, with respect to k, becomes:

$$\mathcal{L}_{\beta_3} \equiv \frac{\vec{j} \cdot \vec{A}}{\varepsilon_0} \tag{5.104}$$

From Eqs. 5.101, 5.103, and 5.104 the $\mathcal{L}_\beta$ part of Lagrange density $\mathcal{L}$ is:

$$\begin{aligned} \mathcal{L}_\beta &= \mathcal{L}_{\beta_1} + \mathcal{L}_{\beta_2} + \mathcal{L}_{\beta_3} \\ &= \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \nabla\phi \cdot \dot{\vec{A}} + \frac{1}{2}c^2 \sum_{k=1}^{k=3} \left[\frac{\partial \vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2 \right] + \frac{\vec{j} \cdot \vec{A}}{\varepsilon_0} \end{aligned} \tag{5.105}$$

Finally, from Eqs. 5.96 and 5.104 for densities $\mathcal{L}_\alpha$and $\mathcal{L}_\beta$, Lagrange density $\mathcal{L} = \mathcal{L}_\alpha + \mathcal{L}_\beta$ is given as:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_\alpha + \mathcal{L}_\beta \\ &= \left(\nabla \cdot \vec{A}\right)\dot{\phi} + \frac{1}{2}\|\nabla\phi\|^2 - \frac{\rho\phi}{\varepsilon_0} \\ &\quad + \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \nabla\phi \cdot \dot{\vec{A}} + \frac{1}{2}c^2 \sum_{k=1}^{k=3} \left[\frac{\partial \vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2 \right] + \frac{\vec{j} \cdot \vec{A}}{\varepsilon_0} \end{aligned} \tag{5.106}$$

Yet, this is a “wrong” Lagrange density; thus, we need to perform the following error-trial-final success procedure (see next section).

5.7.1 *Error-Trial-Final Success*

Insertion of this Lagrange density expression into the Lagrange equation with respect to ϕ (i.e., Eq. 5.92) does not yield Eq. 5.81 but gets:

$$-\nabla^2\phi - \frac{\partial}{\partial t}\left(2\nabla \cdot \vec{A}\right) = \frac{\rho}{\varepsilon_0} \quad \text{(wrong)} \tag{5.107}$$

The appearance of an extra $\left(\nabla \cdot \vec{A}\right)$ is because of the term $\left(\nabla\phi \cdot \dot{\vec{A}}\right)$ of $\mathcal{L}_\beta$ and that is why the Lagrange density given by Eq. 5.106 is not an appropriate one. To resolve this problem, we must look at Eq. 5.90 (i.e., Eq. 5.81) from a different point of view as follows:

$$\nabla \cdot \left(\nabla\phi + \dot{\vec{A}}\right) - \left(-\frac{\rho}{\varepsilon_0}\right) = 0 \tag{5.108}$$

Comparing Eqs. 5.108 and 5.93, we note that the first could be derived from the second if in place of Eq. 5.93 we have the following result:

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0, \quad \frac{\partial\mathcal{L}}{\partial(\nabla\phi)} = \nabla\phi + \dot{\vec{A}}, \quad \frac{\partial\mathcal{L}}{\partial\phi} = -\frac{\rho}{\varepsilon_0} \tag{5.109}$$

So in place of Eqs. 5.94 and Eq. 5.95, respectively, the new form of the equations is as follows:

$$\mathcal{L}'_{\alpha_1} \equiv 0, \quad \mathcal{L}'_{\alpha_2} \equiv \frac{1}{2}\|\nabla\phi\|^2 + \nabla\phi \cdot \dot{\vec{A}}, \quad \mathcal{L}'_{\alpha_3} \equiv \mathcal{L}_{\alpha_3} \equiv -\frac{\rho\phi}{\varepsilon_0} \tag{5.110}$$

$$\mathcal{L}'_\alpha = \mathcal{L}'_{\alpha_1} + \mathcal{L}'_{\alpha_2} + \mathcal{L}'_{\alpha_3} = \frac{1}{2}\|\nabla\phi\|^2 + \left(\nabla\phi \cdot \dot{\vec{A}}\right) - \frac{\rho\phi}{\varepsilon_0} \tag{5.111}$$

Now it is necessary to omit from $\mathcal{L}_{\beta_1}$, Eq. 5.100, the second term $\left(\nabla\phi \cdot \dot{\vec{A}}\right)$ because it appears in $\mathcal{L}'_{\alpha_2}$ (see the second term of Eq. 5.110). Thus, we have in place of Eq. 5.100, the following form:

$$\mathcal{L}'_{\beta_1} \equiv \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 \tag{5.112}$$

Although $\mathcal{L}_{\beta_2}$ and $\mathcal{L}_{\beta_3}$ remain unchanged, as in Eqs. 5.102 and 5.104, we have:

$$\mathcal{L}'_{\beta_2} = \mathcal{L}_{\beta_2} \equiv \frac{1}{2}c^2 \sum_{k=1}^{k=3}\left[\frac{\partial\vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2\right] \tag{5.113}$$

$$\mathcal{L}'_{\beta_3} = \mathcal{L}_{\beta_3} \equiv \frac{\vec{J} \cdot \vec{A}}{\varepsilon_0} \tag{5.114}$$

In place of Eq. 5.105 we now can write:

$$\begin{aligned}\mathcal{L}'_\beta &= \mathcal{L}'_{\beta_1} + \mathcal{L}'_{\beta_2} + \mathcal{L}'_{\beta_3} \\ &= +\frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \frac{1}{2}c^2\sum_{k=1}^{k=3}\left[\frac{\partial\vec{A}}{\partial x_k} \cdot \nabla A_k - \|\nabla A_k\|^2\right] + \frac{\vec{j} \cdot \vec{A}}{\varepsilon_0}\end{aligned} \tag{5.115}$$

Finally, for the new Lagrangian density we have the following in place of Eq. 5.106:

$$\begin{aligned}\mathcal{L}' &= \mathcal{L}'_\alpha + \mathcal{L}'_\beta \\ &= \frac{1}{2}\|\nabla\phi\|^2 + \nabla\phi\cdot\dot{\vec{A}} - \frac{\rho\phi}{\varepsilon_0} \\ &\frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \frac{1}{2}c^2\sum_{k=1}^{k=3}\left[\frac{\partial\vec{A}}{\partial x_k}\cdot\nabla A_k - \|\nabla A_k\|^2\right] + \frac{\vec{j}\cdot\vec{A}}{\varepsilon_0}\end{aligned} \tag{5.116}$$

Density $\mathcal{L}'$ of Eq. 5.116 is obtained from density $\mathcal{L}$ of Eq. 5.106 if we omit the term $(\nabla\cdot\vec{A})\dot{\phi}$. Thus, $\mathcal{L}'$ is *independent* of $\dot{\phi}$.

In the following equations the brace over the left three terms groups that part of density $\mathcal{L}'$, which essentially participates in the production of electromagnetic Eq. 5.81 from the Lagrange equation with respect to ϕ, Eq. 5.92. Although the brace under the right four terms groups that part of density $\mathcal{L}'$, which essentially participates in the production of electromagnetic Eq. 5.83 from the Lagrange equations with respect to A_1, A_2, A_3, Eq. 5.87. Thus, we have:

$$\begin{aligned}\mathcal{L}' &= \overbrace{\frac{1}{2}\|\nabla\phi\|^2 - \frac{\rho\phi}{\varepsilon_0} + \nabla\phi\cdot\dot{\vec{A}}}^{\text{with respect to } \phi} + \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \frac{1}{2}c^2\sum_{k=1}^{k=2}\left[\frac{\partial\vec{A}}{\partial x_k}\cdot\nabla A_k - \|\nabla A_k\|^2\right] + \frac{\vec{j}\cdot\vec{A}}{\varepsilon_0} \\ \mathcal{L}' &= \frac{1}{2}\|\nabla\phi\|^2 - \frac{\rho\phi}{\varepsilon_0} + \underbrace{\nabla\phi\cdot\dot{\vec{A}} + \frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \frac{1}{2}c^2\sum_{k=1}^{k=2}\left[\frac{\partial\vec{A}}{\partial x_k}\cdot\nabla A_k - \|\nabla A_k\|^2\right] + \frac{\vec{j}\cdot\vec{A}}{\varepsilon_0}}_{\text{with respect to } \vec{A}}\end{aligned} \tag{5.117}$$

Note the common term $\left(\nabla\phi\cdot\dot{\vec{A}}\right)$.

Recording the terms in Eq. 5.116 of density $\mathcal{L}'$, we have:

$$\mathcal{L}' = \underbrace{\frac{1}{2}\left\|\dot{\vec{A}}\right\|^2 + \frac{1}{2}\|\nabla\phi\|^2 + \nabla\phi\cdot\dot{\vec{A}}}_{\frac{1}{2}\left\|-\nabla\phi-\frac{\partial\vec{A}}{\partial t}\right\|^2} - \frac{1}{2}c^2\underbrace{\sum_{k=1}^{k=3}\left[\|\nabla A_k\|^2 - \frac{\partial\vec{A}}{\partial x_k}\cdot\nabla A_k\right]}_{\|\nabla\times\vec{A}\|^2} + \frac{1}{\varepsilon_0}(-\rho\phi + \vec{j}\cdot\vec{A}) \tag{5.118}$$

that is,

$$\mathcal{L}' = \frac{1}{2}\left\|-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right\|^2 - \frac{1}{2}c^2\|\nabla\times A\|^2 + \frac{1}{\varepsilon_0}\left(-\rho\phi + \vec{j}\cdot\vec{A}\right) \tag{5.119}$$

or

$$\mathcal{L}' = \frac{\|\vec{E}\|^2 - c^2\|\vec{B}\|^2}{2} + \frac{1}{\varepsilon_0}\left(-\rho\phi + \vec{j}\cdot\vec{A}\right) \tag{5.120}$$

Now if density $\mathcal{L}'$ must have dimensions of energy per unit volume, we define $\mathcal{L}_{em} = \varepsilon_0\mathcal{L}'$; therefore, we can write the following result:

$$\boxed{\mathcal{L}_{em} = \varepsilon_0\left[\frac{\|\vec{E}\|^2 - c^2\|\vec{B}\|^2}{2}\right] - \rho\phi + \vec{j}\cdot\vec{A}} \tag{5.121}$$

keeping in mind that

$$\|\vec{E}\|^2 = \left\|-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right\|^2 = \left\|\dot{\vec{A}}\right\|^2 + \|\nabla\phi\|^2 + 2\left(\nabla\phi\cdot\dot{\vec{A}}\right) \tag{5.122a}$$

$$\|\vec{B}\|^2 = \left\|\vec{\nabla}\times\vec{A}\right\|^2 = \sum_{k=1}^{k=3}\left[\|\nabla A_k\|^2 - \frac{\partial\vec{A}}{\partial x_k}\cdot\nabla A_k\right] \tag{5.122b}$$

The scalar $\left(\|\vec{E}\|^2 - c^2\|\vec{B}\|^2\right)$ is one of the two *Lorentz invariants* (e.g., see subsection in the preceding description of the field); the other one is $\vec{E}\cdot\vec{B}$. The scaler essentially is equal to constant time $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$, where $\varepsilon^{\mu\nu}$ is the anti-symmetric field tensor (see next subsection as well). (More details can be found in Appendix A under the topic Relativity and Electromagnetism.)

On the other hand, the scalar $\left(-\rho\phi + \vec{j}\cdot\vec{A}\right)$ is essentially the inner product $J_\mu A^\mu$ of two 4-vectors in *Minkowski space*. The four-current density $J^\mu = \left(c\rho, \vec{J}\right)$ and the four-potential $A^\mu = \left(\phi/c, \vec{A}\right)$ is a Lorentz invariant scalar too. Therefore, Lagrangian density $\mathcal{L}_{em}$ in Eq. 5.6 is a Lorentz invariant.

5.8 Formal Structure of Maxwell's Theory

Magnetic theory and its set of equations, in essence, now stands before us. To make the connection with its usual three-dimensional formulation in terms of *electric field* $\vec{E}$ and *magnetic field* $\vec{B}$, we define $\vec{E}$ and $\vec{B}$ in each inertial frame by the first of the following equations:

$$
\begin{aligned}
&\mathcal{E}_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{bmatrix} \\
\text{so} \\
&\mathcal{E}^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}
\end{aligned}
\tag{5.123}
$$

which, by making the duality replacement $\vec{E} \to c\vec{B}$ and $c\vec{B} \to \vec{E}$, yields in the following tensor set:

$$
\begin{aligned}
&\mathscr{B}_{\mu\nu} = \begin{bmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & -E_3 & E_2 \\ cB_2 & cE_3 & 0 & -E_1 \\ cB_3 & -E_2 & E_1 & 0 \end{bmatrix} \\
\text{so} \\
&\mathscr{B}^{\mu\nu} = \begin{bmatrix} 0 & cB_1 & cB_2 & -cB_3 \\ E_1 & 0 & -cB_3 & cE_2 \\ E_2 & cB_3 & 0 & -E_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}
\end{aligned}
\tag{5.124}
$$

The two *invariants* of $E^{\mu\nu}$—immediately recognizable as such from their mode of formation—can be expressed as follows:

$$
\begin{cases} X = \dfrac{1}{2}\mathcal{E}_{\mu\nu}\mathcal{E}^{\mu\nu} = -\dfrac{1}{2}\mathscr{B}_{\mu\nu}\mathscr{B}^{\mu\nu} = c^2\left\|\vec{B}\right\|^2 - \left\|\vec{E}\right\|^2 \\ Y = \dfrac{1}{2}\mathscr{B}_{\mu\nu}\mathcal{E}^{\mu\nu} = c\vec{B}\cdot\vec{E} \end{cases}
\tag{5.125}
$$

But if we expand the first equation into the form of *Stueckelberg-Lagrangian density* in the following equation form, we have [1]:

$$
\mathscr{L} = -\frac{\varepsilon c^2}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu - \frac{\gamma\varepsilon c^2}{2}\left(\partial_\mu A^\mu\right)^2 - \frac{\varepsilon c^2 k^2}{2}\left(A_\mu A^\mu\right)
\tag{5.126}
$$

Note that this equation is known as the Lagrangian density equation, or what we call the *More Complete Electromagnetic* (MCE) theory in order to generate a *scalar longitudinal wave* (SLW); more information about such matters is addressed in Chap. 6 of this book.

Equation 5.126 will be written in a general form based on the following argument. That is, the *Helmholtz theorem* (e.g., Griffiths, 2007 [2]) states that any three-dimensional vector (electrical current density $\vec{J}$, in this case) can be uniquely decomposed into circulating- and gradient-driven components, thus $\vec{J} = \nabla\kappa + \vec{\nabla} \times \vec{G}$. Here κ and $\vec{G}$ are scalar and vector functions of spacetime, respectively.

Using the result of Woodside [4] provides a unique decomposition for a smooth *Minkowski field*, and using only *4-vetor geometry* will produce a new form of Maxwell's equations as follows:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{5.127}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{5.128}$$

$$C \equiv \nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{5.129}$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \underbrace{\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}}_{C} = \mu \vec{J} \tag{5.130}$$

$$\vec{\nabla} \cdot \vec{E} + \frac{\partial}{\partial t}\underbrace{\left\{\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right\}}_{C} = \frac{\rho}{\varepsilon} \tag{5.131}$$

$\vec{A}$ and ϕ are the classical vector and scalar potentials, respectively. $\vec{B}$ and $\vec{E}$ are the magnetic and electric fields, respectively, as well. In the preceding set of Maxwell's equations, constant C is a non-zero dynamic field with the *same dimensions* as the magnetic field.

Bear in mind that in classical electrodynamics, $C = 0$ is known as the *Lorentz gauge*. Also, in the previous set of equations ρ and $\vec{J}$ are electric charge density and current charge density, respectively, while ε and μ represent the permittivity and permeability of media, not necessarily of the vacuum, respectively.

The homogeneous equations $\nabla \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \left(\partial \vec{B}/\partial t\right) = 0$ are identical to the classical model, yielding Eqs. 5.129 and 5.130. Using Eqs. 5.127 through 5.131, the most general Lagrangian density for a mass-less, 4-vector field (A^{μ}), which is no more than quadratic in its variables and derivatives, is given by the following equation based on Eq. 2 in Stueckelberg's 1938 article [1]—that is, Stueckelberg-Lagrangian density:

$$\vec{\mathscr{L}} = \frac{\varepsilon c^2}{2}\left[\frac{1}{c^2}\left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right)^2 - \left(\nabla \times \vec{A}\right)^2\right] - \rho\phi + \vec{J} \cdot \vec{A} - \frac{\gamma \varepsilon c^2}{2}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)^2 \tag{5.132}$$

The result we obtained in Eq. 5.126 will be applied to generate a SLW, where we use it as a Lagrangian density equation for what we call the MCE theory.

References

1. F.C.G. Stueckelberg, Interaction forces in electrodynamics and in field theory of nuclear forces. Helv. Phys. Acta. **11**, 299–328 (1938)
2. D. Griffiths, *Introduction to electrodynamics*, 3rd edn. (Prentice Hall, Upper Saddle River, 1999)
3. C.K. Law, Interaction between a moving mirror and radiation pressure: A Hamiltonian formulation. Phys. Rev. A **51**, 2537 (1995)
4. D.A. Woodside, Three-vector and scalar field identities and uniqueness theorems in Euclidean and Minkowski spaces. Am. J. Phys. **77**, 438–446 (2009)

View publication stats